Assessment on Approximation Algorithms

December 20, 2024

Max Marks: 100

Time: 3 Hours

Instructions:

- Attempt all questions. Each question carries equal marks.
- Show all workings for full credit.
- This paper tests advanced topics; innovative approaches are encouraged.
- Use formal mathematical notation wherever applicable.

Questions:

Q1. Set Cover Problem: Prove that the greedy algorithm for the Set Cover problem achieves an approximation ratio of H_d, where H_d is the dth harmonic number, and demonstrate why this ratio is the best possible unless $P = NP$. Additionally, construct a specific instance where the greedy algorithm performs optimally and another instance where it performs poorly.

Q2. Max-Cut Problem: Derive the approximation ratio achieved by the Goemans-Williamson algorithm using semidefinite programming (SDP) for the Max-Cut problem. Discuss the geometric interpretation of the algorithm and provide a detailed proof of the randomized rounding technique.

Q3. Vertex Cover Problem: Consider the Vertex Cover problem and show that any polynomial-time approximation algorithm cannot achieve a ratio better than $2 - \epsilon$ unless the Unique Games Conjecture is false. Use the PCP theorem in your argument.

Q4. Travelling Salesman Problem (TSP): Analyze the Christofides algorithm for metric TSP and prove its approximation ratio. Then, provide an instance of TSP for which no deterministic algorithm can achieve an approximation better than $\frac{3}{2}$.

Q5. Knapsack Problem: Demonstrate the working of the Fully Polynomial-Time Approximation Scheme (FPTAS) for the Knapsack problem. Prove its correctness and discuss the trade-off between the running time and the approximation quality.

Q6. Facility Location Problem: For the uncapacitated facility location problem, derive the dual LP relaxation and show how to construct a 1.61-approximation algorithm using primal-dual schema. Prove its approximation guarantee rigorously.

Q7. Hardness of Approximation: Prove that there exists no polynomial-time approximation algorithm for the Clique problem within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$. Use the gap amplification technique in your proof.

Q8. Approximation in Graph Coloring: Analyze the best-known approximation algorithm for the Minimum Coloring problem and its approximation ratio. Then, discuss why improving this ratio is unlikely based on the hardness results for 3-colorability.

Q9. Approximation Scheme for Planar Graphs: Design and analyze a Polynomial-Time Approximation Scheme (PTAS) for the Planar Maximum Independent Set problem. Prove why the approach does not generalize to general graphs.

Q10. Streaming Algorithms: Discuss the challenges of designing approximation algorithms in a streaming setting. Prove that for the Frequency Moments problem, any streaming algorithm achieving a $(1 + \epsilon)$-approximation for F_k (where $k > 2$) requires $\Omega(n^{1-1/k})$ space.

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